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Residue Expansions for Certain Green's Functions and Resolvent Kernels

A. S. Peters

Prepared under Contract Nonr-285(55) with
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RESIDUE EXPANSIONS FOR CERTAIN GREEN'S FUNCTIONS AND RESOLVENT KERNELS

A. S. Peters

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Abstract

This report is concerned with the expansion of an arbitrary function in terms of the generalized eigenfunctions defined by

$$\frac{d}{dx}\, p(x)\, \frac{d}{dx}\, \psi(x,\lambda) + q(x)\psi - \lambda r(x)\psi = 0 , \qquad 0 < x < 1 ;$$

subject to the boundary conditions

$$\psi_x(0,\lambda) + \beta_0\psi(0,\lambda) = \alpha_0\lambda\psi(0,\lambda) ,$$

$$\psi_x(1,\lambda) + \beta_1\psi(1,\lambda) = \alpha_1\lambda\psi(1,\lambda) .$$

The expansions stem from the Poincaré-Birkhoff formula. It is shown that this formula can be extended so as to yield expansions in terms of the eigenfunctions of

$$\psi(t) = \lambda\left[a_0K(0,t)\psi(0) - a_1K(1,t)\psi(1) + \int_0^1 K(t,x)\psi(x)dx\right]$$

where $K(t,x) = K(x,t)$. The quantities α_0, α_1, a_0, and a_1 are supposed to be real but they are otherwise not restricted.

1. Introduction

A differential equation of the type

$$\frac{d}{dx}\, p(x)\, \frac{d}{dx}\, \psi(x,\lambda) + q(x)\psi - \lambda r(x)\psi = 0 \tag{1.1}$$

to be satisfied for $0 < x < 1$; and subject to the boundary conditions

$$\psi_x(0,\lambda) + \beta_o \psi(0,\lambda) = 0 \tag{1.2}$$

$$\psi_x(1,\lambda) + \beta_1 \psi(1,\lambda) = 0 \ , \tag{1.3}$$

poses what can be called a simple Sturm-Liouville eigenvalue problem provided neither β_o nor β_1 depends on λ. The problem is to find values of the parameter λ for which the system (1.1)-(1.3) possesses solutions which are not identically zero. These values, $\lambda = \lambda_n$, are called eigenvalues and the corresponding functions $\psi_n(x) = \psi(x,\lambda_n)$ are called eigenfunctions. In general, the above system defines a denumerable set of eigenfunctions which satisfy

$$\int_0^1 r(x)\psi_n(x)\psi_m(x)dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \ . \end{cases} \tag{1.4}$$

With respect to the set $\{\psi_n(x)\}$ there is a fundamental expansion problem which is to ascertain if a set of constants $\{c_n\}$ exists

such that the series $\sum_{n=1}^{\infty} c_n \psi_n(x)$ converges in some sense to an arbitrary member $f(x)$ of some class of functions defined for $0 < x < 1$. If $f(x)$ can be represented in the form

$$f(x) = \sum_{n=1}^{\infty} c_n \psi_n(x) \tag{1.5}$$

where the series converges uniformly, then c_k can be fixed by multiplying (1.5) by $\psi_k(x)$ and integrating. This gives $c_k = \int_0^1 r(x) f(x) \psi_k(x) dx$ after we use (1.4). Apart from any special assumption about convergence, the generalized Fourier series

$$\sum_{n=1}^{\infty} \psi_n(x) \int_0^1 r(\xi) f(\xi) \psi_n(\xi) d\xi \tag{1.6}$$

can be studied as a separate entity in order to find how it converges, say to a sum $s(x)$, and how $s(x)$ is related to $f(x)$. It is well known, for example, that if $f(x)$ is piecewise smooth for $0 \leq x \leq 1$ then

$$\tfrac{1}{2} [f(x+0) + f(x-0)] = \sum_{n=1}^{\infty} \psi_n(x) \int_0^1 r(\xi) f(\xi) \psi_n(\xi) d\xi \ .$$

The eigenfunction expansion problem increases in complexity when the set $\{\psi_n(x)\}$ of eigenfunctions is defined by the boundary value problem

$$\frac{d}{dx} p(x) \frac{d}{dx} \psi(x,\lambda) + q(x)\psi - \lambda r(x)\psi = 0 \tag{1.7}$$

$$\psi_x(0,\lambda) + \beta_o\psi(0,\lambda) = \alpha_o\lambda\psi(0,\lambda) \tag{1.8}$$

$$\psi_x(1,\lambda) + \beta_1\psi(1,\lambda) = \alpha_1\lambda\psi(1,\lambda) \ . \tag{1.9}$$

Here, the eigenparameter explicitly appears in the boundary conditions if α_o and α_1 are not simultaneously equal to zero; and the implications of this lead to results substantially different from those indicated above. The inherent character of the expansion problem connected with (1.7)-(1.9) can be appraised to some extent after a brief study of the case defined by

$$\psi_{xx} - \lambda\psi = 0 \tag{1.10}$$

$$\psi_x(0) = 0 \tag{1.11}$$

$$\psi_x(1) = \lambda\psi(1) \ . \tag{1.12}$$

For this case the eigenvalues must satisfy

$$\tanh \sqrt{\lambda_n} = \sqrt{\lambda_n} \tag{1.13}$$

and beginning with $\lambda_o = 0$ they can be ordered with respect to absolute magnitude. The eigenfunctions

$$\psi_n(x) = a_n \cosh x\sqrt{\lambda_n} \ , \qquad n = 0,1,2,... \tag{1.14}$$

constitute a denumerable set such that

$$\psi_n(1)\psi_m(1) - \int_0^1 \psi_n(x)\psi_m(x)dx = 0 \tag{1.15}$$

when $m \neq n$. Suppose that there are functions which can be represented in the form

$$f(x) = c_o + \sum_{n=1}^{\infty} c_n\psi_n(x) \tag{1.16}$$

where the ψ_n's are given by (1.14) and the series converges uniformly for $0 \leq x \leq 1$. Then if we multiply (1.16) by $\psi_k(x)$, integrate; and subtract the integrated series from

$$\psi_k(1)f(1) = c_o\psi_k(1) + \sum_{n=1}^{\infty} c_n\psi_k(1)\psi_n(1)$$

we find, after using (1.15) and (1.13), that

$$c_k = \frac{2\left[\psi_k(1)f(1) - \int_0^1 \psi_k(\xi)f(\xi)d\xi\right]}{a_k^2 \sinh^2\sqrt{\lambda_k}} . \tag{1.17}$$

This suggests the tentative association of $f(x)$ with the series determined by (1.17), that is

$$f(x) \sim c_o + 2\sum_{n=1}^{\infty} \frac{\left[\psi_n(1)f(1) - \int_0^1 \psi_n(\xi)f(\xi)d\xi\right]\psi_n(x)}{a_n^2 \sinh^2\sqrt{\lambda_n}} . \tag{1.18}$$

This association, however, must be rejected because if we choose $f(x) = x^2$ and note that

$$\psi_n(1) - \int_0^1 \xi^2 \psi_n(\xi)d\xi = 0 , \qquad n \geq 1$$

we see that (1.18) forces us to associate x^2 with the constant c_o. We conclude that the set of eigenfunctions (1.14) is inadequate for expansion purposes. This example raises the problem of finding a set of functions $\{\chi_n(x)\}$ which contains the set $\{\psi_n(x)\}$ and admits the expansion

$$f(x) = \sum_{n=0}^{\infty} c_n \chi_n(x) .$$

The example (1.10)-(1.12) also points to the need for a deeper analysis of the expansion problem than the one which depends on the investigation of the convergence of a generalized Fourier series formed from a set of eigenfunctions by using their orthogonality properties.

A basic expansion formula can be obtained by studying the Green's function associated with (1.7)-(1.9). This function $G(x,\xi,\lambda)$ is defined by the equation

$$(1.19) \qquad \frac{d}{dx} p(x)G_x + q(x)G - \lambda r(x)G = \delta(x-\xi) , \qquad 0 < x, \xi < 1$$

subject to the boundary conditions

$$(1.20) \qquad G_x(0,\xi,\lambda) + \beta_o G(0,\xi,\lambda) = \alpha_o \lambda G(0,\xi,\lambda)$$

$$(1.21) \qquad G_x(1,\xi,\lambda) + \beta_1 G(1,\xi,\lambda) = \alpha_1 \lambda G(1,\xi,\lambda) .$$

The right member of (1.19) is the generalized function known as the Dirac delta function. If the parameter λ is regarded as a complex variable, the Cauchy integral formula coupled with the theory of residues can be used to show that when the boundary value problem is subject to certain additional conditions, a function $f(x)$ can be expressed in the form

$$(1.22) \qquad f(x) = \lim_{n\to\infty} \frac{1}{2\pi i} \oint_{\Gamma_n} \int_0^1 G(t,x,\lambda) f(t)\,dt\,d\lambda$$

if $f(x)$ belongs to a restricted class of functions. The contour Γ_n in (1.22) is one which contains the first n+1 eigenvalues defined by (1.7)-(1.9) and the integral along it is equal to a sum of residues determined by the singular points of $G(t,x,\lambda)$ as a function of λ. Thus the formula (1.22) allows the expansion of $f(x)$ into an infinite sum of residues. If but one eigenfunction corresponds to each eigenvalue and if the singular points of G are only simple poles then (1.22) becomes a generalized Fourier series representation for f. The formula (1.22) was noted by Poincaré [1] during some work on a special problem in partial differential equations. Then Birkhoff [2] proved the formula for an ordinary n^{th} order boundary value problem subject to certain regularity assumptions and with the eigenvalue parameter absent from the boundary conditions. Later, Tamarkin [3] showed that the formula is valid for a wide class of boundary value problems with the parameter present in the boundary conditions. Since then, the formula has been proved by Wilder [4], Langer [5], Rasulov [6] and others

under less restrictive conditions than those used by Tamarkin.

The primary purpose of this report is to present a proof of (1.22) as it is related to (1.19)-(1.21) where the real constants in the boundary conditions are otherwise not restricted. The proof is believed to be new because, unlike any other the author has seen, it does not depend on explicit asymptotic estimates of the behavior of the eigenvalues and eigenfunctions defined by (1.7)-(1.9) as $\lambda \longrightarrow \infty$. Since asymptotic evaluations are not used the proof can be adapted to cover expansion problems which arise in connection with other functional equations and particularly those for which it is not easy to find the asymptotic behavior of the elements of the pertinent expansion. This is illustrated in Section 5 where the integral equation

$$R(t,\xi,\lambda) = K(t,\xi) + \lambda \left\{ \begin{array}{l} \displaystyle\int_0^1 K(x,t)R(x,\xi,\lambda)dx \\ \\ \quad - a_1 K(1,t)R(1,\xi,\lambda) \\ \\ \quad + a_o K(0,t)R(0,\xi,\lambda) \end{array} \right\} \tag{1.23}$$

with a symmetric kernel

$$K(x,t) = K(t,x)$$

is studied briefly. For functions $f(x)$ expressible in the form

$$f(x) = \int_0^1 K(t,x)F(t)dt$$

we prove that the resolvent kernel $R(t,\xi,\lambda)$ satisfies

$$\lim_{n\to\infty} \frac{1}{2\pi i} \oint_{\Gamma_n} \frac{\int_0^1 F(t)R(t,x,z)dt}{z} dz = 0 \tag{1.24}$$

where Γ_n contains the first n+1 eigenvalues associated with (1.23). The result (1.24) leads to the expression of f(x) as an infinite sum of residues.

We admit the ranges

$$-\infty < \alpha_o, a_o < \infty$$

and

$$-\infty < \alpha_1, a_1 < \infty$$

for both the Green's function $G(x,\xi,\lambda)$ and the resolvent kernel $R(x,\xi,\lambda)$. These ranges are not usually considered for the Green's function; and as far as we know they have not been considered with respect to $R(x,\xi,\lambda)$.

2. Some Properties of the Green's Function

In preparation for later developments, we collect in this section some of the relations which involve the Green's function $G(x,\xi,\lambda)$. This function is defined by the equation

$$LG - \lambda r(x)G = \delta(x-\xi) , \qquad 0 < x, \xi < 1 \tag{2.1}$$

subject to the boundary conditions

$$G_x(0,\xi,\lambda) + \beta_o G(0,\xi,\lambda) = \alpha_o \lambda G(0,\xi,\lambda) \tag{2.2}$$

and

$$G_x(1,\xi,\lambda) + \beta_1 G(1,\xi,\lambda) = \alpha_1 \lambda G(1,\xi,\lambda) \ . \tag{2.3}$$

The symbol L is used to denote the operation defined by

$$L\Omega \equiv \frac{d}{dx}\, p(x)\, \frac{d}{dx}\, \Omega(x) + q(x)\Omega(x) \ ;$$

and with respect to this we have the fundamental and useful identity

$$\int_\xi^t \chi L\Omega dx = \left| p(\chi\Omega_x - \Omega\chi_x) \right|_\xi^t + \int_\xi^t \Omega L\chi dx \ . \tag{2.4}$$

We suppose that $p(x)$ has a continuous derivative and that $p(x) \geq 0$ for $0 \leq x \leq 1$. We also suppose that each of $q(x)$ and $r(x)$ is continuous for $0 \leq x \leq 1$ while $r(x)$ is non-negative. Later on, the constants in the boundary conditions will be allowed to assume any real value; but <u>for the time being we take</u> $\alpha_1 \leq 0$ <u>and</u> $\alpha_o \geq 0$.

The symbol $\delta(x-\xi)$ on the right-hand side of (2.1) is used in what follows to denote the generalized function such that when it is multiplied by a piecewise continuous function $F(x)$ with a finite number of finite jumps and formally integrated the result means

$$\int_a^b \delta(x-\xi)F(x)dx = \frac{1}{2}\,[F(\xi+0) + F(\xi-0)]$$

when $a < \xi < b$.

If $G(x,\xi,\lambda)$ exists for $\lambda = z$ and $\lambda = \sigma$; and if we set $\Omega = G(x,t,z)$, $\chi = G(x,\xi,\sigma)$ in (2.4), this identity gives

$$(2.5)\qquad (z-\sigma)\left\{\begin{array}{l}\displaystyle\int_0^1 r(x)G(x,t,z)G(x,\xi,\sigma)dx \\ \\ \quad - p(1)\alpha_1 G(1,\xi,\sigma)G(1,t,z) \\ \\ \quad + p(0)\alpha_0 G(0,\xi,\sigma)G(0,t,z)\end{array}\right\} = G(\xi,t,z) - G(t,\xi,\sigma)\ .$$

This leads to some interesting consequences. If we set $z = \sigma$ in (2.5) we have

$$G(\xi,t,z) = G(t,\xi,z)$$

which shows that G is symmetric in t and ξ. If we divide (2.5) by $z-\sigma$ and then take the limit as $\sigma \longrightarrow z$ we have

$$(2.6)\qquad \frac{\partial}{\partial z} G(t,\xi,z) = \left\{\begin{array}{l}\displaystyle\int_0^1 r(x)G(x,t,z)G(x,\xi,z)dx \\ \\ \quad - p(1)\alpha_1 G(1,t,z)G(1,\xi,z) \\ \\ \quad + p(0)\alpha_0 G(0,t,z)G(0,\xi,z)\end{array}\right\}\ .$$

If we take $\sigma = 0$ in (2.5) we see that $G(t,\xi,z)$ must satisfy the integral equation

$$(2.7)\qquad G(t,\xi,z) = g(t,\xi) + z\left\{\begin{array}{l}\displaystyle\int_0^1 r(x)g(x,\xi)G(x,t,z)dx \\ \\ \quad - p(1)\alpha_1 g(1,\xi)G(1,t,z) \\ \\ \quad + p(0)\alpha_0 g(0,\xi)G(0,t,z)\end{array}\right\}$$

provided of course that

$$G(x,\xi,0) = g(x,\xi)$$

exists. When it exists $g(x,\xi)$ must satisfy

$$Lg = \delta(x-\xi) \tag{2.8}$$

$$g_x(0,\xi) + \beta_0 g(0,\xi) = 0 \tag{2.9}$$

$$g_x(1,\xi) + \beta_1 g(1,\xi) = 0 \ . \tag{2.10}$$

Operations of the type which appear within the braces in equation (2.7) occur frequently in the sequel and it is convenient to have a special symbol for them. We therefore define $Q[\Omega,\chi]$ to mean

$$Q[\Omega(x),\chi(x)] = \left\{ \begin{array}{l} \displaystyle\int_0^1 r(x)\Omega(x)\chi(x)dx \\ \\ \quad - p(1)\alpha_1\Omega(1)\chi(1) \\ \\ \quad + p(0)\alpha_0\Omega(0)\chi(0) \end{array} \right\} \ . \tag{2.11}$$

Equations (2.5) and (2.7) show that in terms of the operator Q, the Green's function $G(t,\xi,z)$ must satisfy the functional equations

$$G(t,\xi,z) - G(t,\xi,\sigma) = (z-\sigma)Q[G(x,t,z),G(x,\xi,\sigma)] \tag{2.12}$$

and

$$G(t,\xi,z) = g(t,\xi) + zQ[g(x,\xi).G(x,t,z)] \ . \tag{2.13}$$

For later use let us note here that our assumptions about $p(x)$, $r(x)$, α_o and α_1 show that $Q[\Omega(x),\Omega(x)]$ is never negative. Therefore for all s

$$Q[s\Omega+\chi, s\Omega+\chi] = s^2Q[\Omega,\Omega] + 2sQ[\Omega,\chi] + Q[\chi,\chi] \geq 0 \ ;$$

and in order to satisfy this the discriminant of the quadratic in s must be less than or equal to zero. This implies an inequality of the Schwartz type, namely

$$Q^2[\Omega,\chi] \leq Q[\Omega,\Omega]Q[\chi,\chi] \ . \tag{2.14}$$

If $u(x,\lambda)$ and $v(x,\lambda)$ are linearly independent solutions of

$$L\phi - \lambda r\phi = 0$$

such that

$$u_x(0,\lambda) + \beta_o u(0,\lambda) = \alpha_o \lambda u(0,\lambda)$$

and

$$v_x(1,\lambda) + \beta_1 v(1,\lambda) = \alpha_1 \lambda v(1,\lambda)$$

then the identity (2.4) can be used to prove that

$$G(t,\xi,\lambda) = \begin{cases} \dfrac{u(t,\lambda)v(\xi,\lambda)}{p(t)[u(t,\lambda)v_t(t,\lambda) - v(t,\lambda)u_t(t,\lambda)]} \ , & 0 \leq t \leq \\ \\ \dfrac{v(t,\lambda)u(\xi,\lambda)}{p(t)[u(t,\lambda)v_t(t,\lambda) - v(t,\lambda)u_t(t,\lambda)]} \ , & \xi \leq t \leq \end{cases} \tag{2.15}$$

Each of $u(x,\lambda)$ and $v(x,\lambda)$ is an entire function of λ regarded as a complex variable. The denominator in (2.15) is independent of t; it is a function of λ only. Furthermore, it can be shown that the denominator in (2.15) vanishes if and only if $\lambda = \lambda_n$ is an eigenvalue of the homogeneous system associated with (2.1)-(2.3), namely

$$L\psi_n - \lambda_n r\psi_n = 0 \tag{2.16}$$

$$\psi_{nx}(0) + \beta_o\psi_n(0) = \alpha_o\lambda_n\psi_n(0) \tag{2.17}$$

$$\psi_{nx}(1) + \beta_1\psi_n(1) = \alpha_1\lambda_n\psi_n(1) \ . \tag{2.18}$$

It is easy to show that all the eigenvalues are real and that

$$Q[\psi_n(x),\psi_m(x)] = 0 \ , \qquad m \neq n \ .$$

The result (2.15) shows that we can express $G(x,\xi,\lambda)$ in the form

$$G(x,\xi,\lambda) = \frac{\theta(x,\xi,\lambda)}{\omega(\lambda)} \ , \tag{2.19}$$

a ratio of two entire functions of λ where the zeros of $\omega(\lambda)$ are just the eigenvalues of (2.16)-(2.18). The function $\theta(x,\xi,\lambda)$ must satisfy

$$L\theta - \lambda r\theta = \omega(\lambda)\delta(x-\xi) \tag{2.20}$$

$$\theta_x(0,\xi,\lambda) + \beta_o\theta(0,\xi,\lambda) = \alpha_o\lambda\theta(0,\xi,\lambda) \tag{2.21}$$

and

$$\theta_x(1,\xi,\lambda) + \beta_1\theta(1,\xi,\lambda) = \alpha_1\lambda\theta(1,\xi,\lambda) \ . \tag{2.22}$$

If $\lambda = \lambda_n$, the system (2.20)-(2.22) reduces to (2.16)-(2.18) which does not depend on ξ. Consequently, there is no loss of generality if we take

$$\theta(x,\xi,\lambda_n) = \psi_n(x) \ . \tag{2.23}$$

The residue of $G(x,\xi,\lambda)$ at $\lambda = \lambda_n$ depends on $\omega'(\lambda)$ which can be found by differentiating (2.20)-(2.22) with respect to λ. This gives

$$L\theta_\lambda - \lambda r\theta_\lambda - r\theta = \omega'(\lambda)\delta(x-\xi) \tag{2.24}$$

$$\theta_{x\lambda}(0,\xi,\lambda) + \beta_o\theta_\lambda(0,\xi,\lambda) = \alpha_o\lambda\theta_\lambda(0,\xi,\lambda) + \alpha_o\theta(0,\xi,\lambda) \tag{2.25}$$

$$\theta_{x\lambda}(1,\xi,\lambda) + \beta_1\theta_\lambda(1,\xi,\lambda) = \alpha_1\lambda\theta_\lambda(1,\xi,\lambda) + \alpha_1\theta(1,\xi,\lambda) \ . \tag{2.26}$$

If we multiply (2.24) by θ, integrate and then set $\lambda = \lambda_n$, a computation shows that

$$\omega'(\lambda_n)\psi_n(\xi) = - Q[\psi_n,\psi_n] \tag{2.27}$$

$$= - \left\{ \begin{array}{l} \displaystyle\int_0^1 r(x)\psi_n^2(x)dx \\ \\ \quad - \alpha_1 p(1)\psi_n^2(1) \\ \\ \quad + \alpha_o p(0)\psi_n^2(0) \end{array} \right\} \quad .$$

Since the quantity $Q[\psi_n,\psi_n]$ is positive, $\omega'(\lambda_n)$ does not vanish and it follows that λ_n is a simple zero of $\omega(\lambda)$. If we please we can adjust ψ_n with a multiplicative factor so that

$$Q[\psi_n,\psi_n] = 1 \ .$$

Then

$$\frac{1}{\omega'(\lambda_n)} = - \psi_n(\xi) \ . \tag{2.28}$$

3. Function Theoretic Analysis of $G(x,\xi,\lambda)$.

If λ is a regular value, that is, a value for which $G(x,\xi,\lambda)$ exists, the Cauchy integral formula allows us to write

$$G(x,\xi,\lambda) = \frac{\theta(x,\xi,\lambda)}{\omega(\lambda)} = \frac{1}{2\pi i} \oint_\Gamma \frac{\theta(x,\xi,z)dz}{(z-\lambda)\omega(z)} \tag{3.1}$$

where Γ contains λ but no zero of $\omega(z)$. If Γ is deformed into a circle C of radius ρ which has its center at the origin and which contains λ and the first m+1 zeros of $\omega(z)$, the Green's function

can be expressed in the expanded form

$$(3.2)\qquad G(x,\xi,\lambda) = -\sum_{n=0}^{m} \frac{1}{2\pi i} \oint_{C_n} \frac{G(x,\xi,\lambda)dz}{z-\lambda} + \frac{1}{2\pi i} \oint_{C} \frac{G(x,\xi,z)dz}{z-\lambda}$$

where C_n is a circle with center at λ_n containing neither no other zero of $\omega(z)$ nor λ. Each term in the sum is the residue of $G(x,\xi,z)/(z-\lambda)$ at $z = \lambda_n$ and by using (2.25) we have

$$\frac{1}{2\pi i} \oint_{C_n} \frac{G(x,\xi,z)dz}{z-\lambda} = \frac{1}{2\pi i} \oint_{C_n} \frac{\theta(x,\xi,z)dz}{(z-\lambda)\omega(z)}$$

$$= \frac{\theta(x,\xi,\lambda_n)}{(\lambda_n-\lambda)\omega'(\lambda_n)} = \frac{\psi_n(x)\psi_n(\xi)}{\lambda-\lambda_n}$$

so that

$$(3.3)\qquad G(x,\xi,\lambda) = \sum_{n=0}^{m} \frac{\psi_n(x)\psi_n(\xi)}{\lambda_n-\lambda} + \frac{1}{2\pi i} \oint_{C} \frac{G(x,\xi,z)dz}{z-\lambda} .$$

Let $\zeta(x,\xi,\lambda)$ be defined by

$$(3.4)\qquad \zeta(x,\xi,\lambda) = \frac{1}{2\pi i} \oint_{C} \frac{G(x,\xi,z)dz}{z-\lambda} .$$

This function is analytic for $|\lambda| < \rho$ and its power series expansion is

$$(3.5)\qquad \zeta(x,\xi,\lambda) = \frac{1}{\lambda} \sum_{j=1}^{\infty} \zeta_j(x,\xi)\lambda^j$$

where

$$(3.6) \qquad \zeta_j(x,\xi) = \frac{1}{2\pi i} \oint_C \frac{G(x,\xi,z)dz}{z^j} \ .$$

If we use equation (2.13), each ζ_j beginning with ζ_2 can be expressed in terms of its predecessor. The formula is

$$(3.7) \qquad \zeta_j(t,\xi) = \frac{1}{2\pi i} \oint_C \frac{\{g(t,\xi) + zQ[g(x,t),G(x,\xi,z)]\}dz}{z^j} \ ,$$

and for $j \geq 2$

$$(3.8) \qquad \zeta_j(t,\xi) = Q[g(x,t),\zeta_{j-1}(x,\xi)] \ .$$

Another important relation between the coefficients comes from applying (2.12) to $Q[\zeta_\ell(x,\xi),\zeta_m(x,\xi)]$. It is

$$(3.9) \quad Q[\zeta_\ell(x,\xi),\zeta_m(x,\xi)]$$

$$= \frac{1}{2\pi i} \oint_C \frac{1}{\sigma^\ell} \cdot \frac{1}{2\pi i} \oint_C \frac{Q[G(x,\xi,z),G(x,\xi,\sigma)]dzd\sigma}{z^m}$$

$$= \frac{1}{2\pi i} \oint_C \frac{1}{\sigma^\ell} \cdot \frac{1}{2\pi i} \oint_C \frac{[G(\xi,\xi,z) - G(\xi,\xi,\sigma)]dzd\sigma}{z^m(z-\sigma)}$$

$$= \zeta_{m+\ell}(\xi,\xi) \ , \qquad m,\ell \geq 1 \ .$$

The relation (3.9) is valid for $m \geq 1$ with $\ell \geq 1$. It shows that the dependence of

$$Q[\zeta_\ell(x,\xi),\zeta_m(x,\xi)] = \zeta_{m+\ell}(\xi,\xi) \equiv I_{m+\ell}(\xi)$$

on $\ell \geq 1$, and $m \geq 1$ is characterized by the sum of the subscripts.

If it can be shown that $\zeta(x,\xi,\lambda) \longrightarrow 0$ as $\rho \longrightarrow \infty$ we have from (3.3) that

$$G(x,\xi,\lambda) = \sum_{n=0}^{\infty} \frac{\psi_n(x)\psi_n(\xi)}{\lambda_n - \lambda}$$

which is a partial fraction expansion for $G(x,\xi,\lambda)$. Our initial interest of course is not so much in this as it is in the behavior of $\zeta_1(x,\xi)$ as $\rho \longrightarrow \infty$. For the purpose of studying this behavior we are going to use an extension of an idea due to Kneser [7]. The extension involves the determination of bounds for the ratios of successive coefficients of the series

$$(3.10) \qquad \frac{\lambda}{2} Q[\zeta_1(x,\xi),\zeta(x,\xi,\lambda) + \zeta(x,\xi,-\lambda)]$$

$$= \sum_{j=1}^{\infty} Q[\zeta_1(x,\xi),\zeta_{2j-1}(x,\xi)]\lambda^{2j-1}$$

$$= \sum_{j=1}^{\infty} \zeta_{2j}(\xi,\xi)\lambda^{2j-1} = \sum I_{2j}(\xi)\lambda^{2j-1}$$

whose radius of convergence ρ_o is such that $\rho - \varepsilon_1 < \rho_o$ where $0 < \varepsilon_1 \leq \rho$.

It follows from the inequality (2.14) that

$$Q^2[\zeta_{j-1}(x,\xi),\zeta_{j+1}(x,\xi)] \leq Q[\zeta_{j-1},\zeta_{j-1}] \cdot Q[\zeta_{j+1},\zeta_{j+1}]$$

and hence for $j \geq 2$

$$\zeta_{2j}^2(\xi,\xi) \leq \zeta_{2j-2}(\xi,\xi) \cdot \zeta_{2j+2}(\xi,\xi)$$

or

$$I_{2j}^2(\xi) \leq I_{2j-2}(\xi) I_{2j+2}(\xi) \ . \tag{3.11}$$

This shows that $I_{2j}(\xi)$ is either zero for all values of $j \geq 2$ or it is positive for all values of $j \geq 2$. If for any $j \geq 2$, $I_{2j}(\xi) = 0$ then

$$I_{2j+2}^2(\xi) \leq I_{2j}(\xi) I_{2j+1}(\xi)$$

and

$$I_{2j+2}(\xi) = 0 \ .$$

Also

$$I_{2j-2}^2(\xi) \leq I_{2j-4}(\xi) I_{2j}(\xi)$$

and hence

$$I_{2j-2}(\xi) = 0 \ .$$

Therefore if $I_{2j}(\xi) = 0$, $j \geq 2$, it follows that $I_4(\xi) = 0$. On the other hand if $I_{2j}(\xi) > 0$ we can write (3.11) in the form

$$\frac{I_{2j}(\xi)}{I_{2j-2}(\xi)} \leq \frac{I_{2j+2}(\xi)}{I_{2j}(\xi)} \ , \qquad j \geq 2 \ . \tag{3.12}$$

Now the radius of convergence of (3.10) is given by

$$\rho_o^2 = \lim_{j \to \infty} \frac{I_{2j}(\xi)}{I_{2j+2}(\xi)} .$$

Since ρ_o can be taken as large as we please this implies that for all $j \geq N$

$$\frac{I_{2j+2}(\xi)}{I_{2j}(\xi)} < \varepsilon .$$

Then by virtue of the inequality (3.12) it follows that for all $k \geq 2$

$$I_{2k} < \varepsilon I_{2k-2}$$

from which

$$\begin{aligned} I_{2k} &< \varepsilon^2 I_{2k-4} \\ &\vdots \quad\quad \vdots \\ I_{2k} &< \varepsilon^{k-1} I_2 \end{aligned} \tag{3.13}$$

and in particular

$$I_4(\xi) < \varepsilon I_2(\xi) . \tag{3.14}$$

We have now shown that either (3.14) holds or else $I_4(\xi) = 0$.

The quantity $I_4(\xi)$ is

$$I_4(\xi) = Q[\zeta_2(x,\xi), \zeta_2(x,\xi)] = \zeta_4(\xi,\xi)$$

or, by (3.8),

$$I_4(\xi) = Q[g(x,\xi), \zeta_3(x,\xi)] .$$

Hence, as a consequence of the inequality (2.14) and the results in the last paragraph, we find

$$I_4^2(\xi) \leq Q[g(x,\xi),g(x,\xi)]I_6(\xi)$$

$$I_4^2(\xi) \leq \varepsilon Q[g,g]I_4(\xi)$$

$$I_4(\xi) \leq \varepsilon Q[g,g] \ .$$

It is now evident that if $I_4(\xi) = 0$, or if (3.14) holds then

$$\lim_{\rho \to \infty} \zeta_2(x,\xi) \equiv \lim_{\rho \to \infty} \frac{1}{2\pi i} \oint_C \frac{G(x,\xi,z)dz}{z^2}$$

is a function which is zero almost everywhere for $0 \leq x, \xi \leq 1$. Such a function is called a null function. By definition, two functions are equivalent if their difference is a null function. Thus, $\lim_{\rho \to \infty} \zeta_2$ is equivalent to 0 and this can be symbolized by writing

$$\lim_{\rho \to \infty} \zeta_2(x,\xi) =^{o} 0 \ .$$

We also have the expansion

$$(3.15)\quad \zeta_2(x,\xi) = \frac{1}{2\pi i}\oint_C \frac{G(x,\xi,z)dz}{z^2}$$

$$= \frac{1}{2\pi i}\oint_{C_o} \frac{G(x,\xi,z)dz}{z^2} + \sum_{n=1}^{m} \frac{1}{2\pi i}\oint_{C_n} \frac{\theta(x,\xi,z)dz}{z^2\omega(z)}$$

$$= \frac{1}{2\pi i}\oint_{C_o} \frac{G(x,\xi,z)dz}{z^2} - \sum_{n=1}^{m} \frac{\psi_n(x)\psi_n(\xi)}{\lambda_n^2}$$

where C_o encloses the origin, which could be a zero of $\omega(z)$, but no other zero of $\omega(z)$. We know that

$$(3.16)\qquad \zeta_2(\xi,\xi) = Q[\zeta_1(x,\xi),\zeta_1(x,\xi)] \geq 0 .$$

Therefore

$$\frac{1}{2\pi i}\oint_{C_o} \frac{G(\xi,\xi,z)dz}{z^2} \geq \sum_{n=1}^{m} \frac{\psi_n^2(\xi)}{\lambda_n^2}$$

which means that

$$\sum_{n=1}^{\infty} \frac{\psi_n^2(\xi)}{\lambda_n^2}$$

converges. From this we have

$$\sum_{n=\ell}^{m} \frac{\psi_n(x)\psi_n(\xi)}{\lambda_n^2} \leq \sum_{n=\ell}^{m} \left|\frac{\psi_n(x)}{\lambda_n}\right|\left|\frac{\psi_n(\xi)}{\lambda_n}\right|$$

$$\leq \sqrt{\sum_{n=\ell}^{m} \frac{\psi_n^2(x)}{\lambda_n^2} \cdot \sum_{n=\ell}^{m} \frac{\psi_n^2(\xi)}{\lambda_n^2}}$$

$$\leq \sqrt{\frac{1}{2\pi i}\oint_{C_o} \frac{G(x,x,z)dz}{z^2} \cdot \sum_{n=\ell}^{m} \frac{\psi_n^2(\xi)}{\lambda_n^2}} ,$$

and since

$$\frac{1}{2\pi i} \oint_{C_o} \frac{G(x,x,z)dz}{z^2}$$

is bounded, while

$$\sum_{n=\ell}^{m} \frac{\psi_n^2(\xi)}{\lambda_n^2}$$

can be made as small as we please by choosing ℓ sufficiently large; we conclude that

$$\sum_{n=1}^{\infty} \frac{\psi_n(x)\psi_n(\xi)}{\lambda_n^2}$$

converges absolutely and uniformly. We deduce from this, and the continuity of the terms in (3.15), that $\zeta_2(x,\xi)$ converges to a continuous function as $\rho \longrightarrow \infty$. This function is already known to be a null function. Hence, from (3.16)

$$Q[\zeta_1(x,\xi),\zeta_1(x,\xi)] < \varepsilon \tag{3.17}$$

for ρ sufficiently large.

The inequality (3.17) proves that by taking ρ large enough we can make $\zeta_1(x,\xi)$ as small as we please almost everywhere. In other words,

$$\lim_{\rho \longrightarrow \infty} \zeta_1(x,\xi) =^{o} 0 \tag{3.18}$$

while

(3.19) $$\lim_{\rho\to\infty} \zeta_1(0,\xi) = 0 \; ; \qquad \lim_{\rho\to\infty} \zeta_1(1,\xi) = 0 \; .$$

The result (3.18), with (2.13) gives

$$\lim_{\rho\to\infty} \frac{1}{2\pi i} \oint_C \frac{G(x,\xi,z)dz}{z}$$

$$= \lim_{\rho\to\infty} \frac{1}{2\pi i} \oint_C \left\{ \frac{g(x,\xi)}{z} + Q[g(t,\xi),G(t,x,z)] \right\} dz$$

$$=^{\circ} 0$$

which yields

(3.20) $$g(x,\xi) =^{\circ} \sum_{n=0}^{\infty} \frac{\psi_n(x)\psi_n(\xi)}{\lambda_n} \; .$$

This means that $g(x,\xi)$ differs from the series in (3.20) by at most a null function. However, (3.19) shows that for the end points the equivalence sign in (3.20) can be replaced by the equality sign, that is

(3.21) $$g(0,\xi) = \sum_{n=0}^{\infty} \frac{\psi_n(0)\psi_n(\xi)}{\lambda_n}$$

and

(3.22) $$g(1,\xi) = \sum_{n=0}^{\infty} \frac{\psi_n(1)\psi_n(\xi)}{\lambda_n} \; .$$

The result (3.18) also gives

$$\frac{1}{2\pi i}\oint_C \frac{\int_0^1 F(x)G(x,\xi,z)dxdz}{z} = \left| \int_0^1 F(x)\zeta_1(x,\xi)dx \right|$$

$$\leq \sqrt{\int_0^1 F^2(x)dx \int_0^1 \zeta_1^2(x,\xi)dx}$$

$$< \varepsilon ,$$

where $F(x)$ is any function such that $\int_0^1 F^2(x)dx$ exists; and this establishes the basic result

$$\lim_{\rho\to\infty} \frac{1}{2\pi i}\oint_C \frac{\int_0^1 F(x)G(x,\xi,z)dxdz}{z} = 0 . \tag{3.23}$$

If we use (2.13), the limit relation (3.23) becomes

$$\lim_{\rho\to\infty} \frac{1}{2\pi i}\oint_C \left\{ \frac{\int_0^1 F(x)g(x,\xi)dx}{z} + \int_0^1 F(x)Q[g(t,x),G(t,\xi,z)]dx \right\} dz = 0 . \tag{3.24}$$

For a function $f(\xi)$ which can be represented in the form

$$f(\xi) = \int_0^1 F(x)g(x,\xi)dz \tag{3.25}$$

where $F(x)$ is such that

$$\lim_{\rho \to \infty} \int_0^1 F(x)\zeta_1(x,\xi)dx = 0 \tag{3.26}$$

the relation (3.24) shows that

$$f(\xi) + \lim_{\rho \to \infty} \frac{1}{2\pi i} \oint_C Q[G(t,\xi,z),f(t)]dz = 0 \ . \tag{3.27}$$

This is an extension of the Poincaré-Birkhoff formula. Under our present assumptions about $p(x)$, $r(x)$, α_o and α_1 the expanded form of (3.27) is

$$f(\xi) = \sum_{n=0}^{\infty} Q[\psi_n(x),f(x)]\psi_n(\xi) \ . \tag{3.28}$$

A twice differentiable function $f(x)$ such that $[f''(x)]^2$ is integrable is an example of a function for which the expansion (3.28) is valid.

Under some circumstances it can be proved that $\zeta_1(x,\xi)$ converges to a continuous function or even one that is differentiable. When either of these possibilities occur our results can of course be sharpened. However, we forego a study of such cases because we are interested here only in presenting an expansion theorem which holds both for the Green's function $G(x,\xi,\lambda)$ and the resolvent kernel $R(x,\xi,\lambda)$ of an integral equation which

generalizes the type (2.13) which $G(x,\xi,\lambda)$ satisfies. This integral equation will be studied later.

Our immediate object now is to prove in the next section that (3.23) continues to hold when we relax the restrictions which have so far been placed on α_o and α_1.

4. Removal of Restrictions

In Section 3 we have assumed that the real constants α_o and α_1 are such that $\alpha_o \geq 0$ while $\alpha_1 \leq 0$. We can free ourselves from these restrictions with respect to sign by using the technique described in what follows. In order to illustrate the procedure without invoking lengthy expressions, let us suppose that α_o is subject to the same restriction imposed above while α_1 is real but otherwise unrestricted. In other words, let us consider the Green's function $\mathcal{G}(x,\xi,\lambda)$ defined by the system

$$L\mathcal{G}(x,\xi,\lambda) - \lambda r(x)\mathcal{G} = \delta(x-\xi) \tag{4.1}$$

$$\mathcal{G}_x(0,\xi,\lambda) + \beta_o \mathcal{G}(0,\xi,\lambda) = \alpha_o \lambda \mathcal{G}(0,\xi,\lambda) \tag{4.2}$$

$$\mathcal{G}_x(1,\xi,\lambda) + \beta_1 \mathcal{G}(1,\xi,\lambda) = \alpha \lambda \mathcal{G}(1,\xi,\lambda) . \tag{4.3}$$

We contemplate allowing α to be either negative, zero or positive.

In order to study $\mathcal{G}(x,\xi,\lambda)$ let us introduce the Green's function $G^*(x,\xi,\lambda)$ which satisfies the system

$$LG^*(x,\xi,\lambda) - \lambda r(x)G^* = \delta(x-\xi) \tag{4.4}$$

$$G_x^*(0,\xi,\lambda) + \beta_o G^*(0,\xi,\lambda) = \alpha_o \lambda G^*(0,\xi,\lambda) \tag{4.5}$$

$$G_x^*(1,\xi,\lambda) + \beta_1 G^*(1,\xi,\lambda) = 0 \ . \tag{4.6}$$

In terms of $G^*(x,\xi,\lambda)$ the function $\mathcal{G}(x,\xi,\lambda)$ is given by

$$\mathcal{G}(t,\xi,\lambda) = G^*(t,\xi,\lambda) - \frac{\alpha p(1)\lambda G^*(t,1,\lambda)G^*(\xi,1,\lambda)}{1+\alpha p(1)\lambda G^*(1,1,\lambda)} \ . \tag{4.7}$$

Let the center of the circular contour C be at the origin and let its radius ρ be chosen so that C avoids the singularities of $\mathcal{G}(t,\xi,\lambda)$ and $G^*(t,\xi,z)$; but contains the first m+1 poles of $G^*(t,\xi,z)$. In order to proceed we need to know something about the behavior of $\mathcal{G}(t,\xi,z)$ for z on C as $\rho \longrightarrow \infty$. We begin by considering $zG^*(1,1,z)$ which appears in the expression (4.7). For finite ρ, $zG^*(1,1,z)$ is analytic on C and

$$\frac{1}{2\pi i} \oint_C zG^*(1,1,z)\,\frac{dz}{z} = -\sum_{n=0}^{m} [\psi_n^*(1)]^2 \ .$$

Suppose $zG^*(1,1,z)$ remains bounded for z on C as $\rho \longrightarrow \infty$. If this is so then

$$\sum_{n=0}^{\infty} [\psi_n^*(1)]^2$$

converges; and an extension of the Riesz-Fischer theorem shows that there exists a function $\mu(x)$ such that $\mu^2(x)$ is integrable in

the Lebesgue sense and such that

$$\mu(x) =^{o} \sum_{0}^{\infty} \psi_n^*(1)\psi_n^*(x) \tag{4.8}$$

where

$$Q^*[\mu(x),\psi_n^*(x)] = \int_0^1 r(x)\mu(x)\psi_n^*(x)dx + \alpha_o p(0)\mu(0)\psi_n^*(0) \tag{4.9}$$

$$= \psi_n^*(1) .$$

On the other hand, we know from Section 3 that

$$G^*(x,\xi,0) = g(x,\xi) =^{o} \sum_{n=0}^{\infty} \frac{\psi_n^*(\xi)\psi_n^*(x)}{\lambda_n} \tag{4.10}$$

while (3.28) shows that

$$\nu(\xi) = \int_0^1 r(x)g(x,\xi)\mu(x)dx = \sum_{n=0}^{\infty} Q^*[\psi_n^*(x),\nu(x)]\psi_n^*(\xi) \tag{4.11}$$

$$= \sum_{n=0}^{\infty} \frac{\psi_n^*(\xi) \int_0^1 r(t)\mu(t)\psi_n^*(t)dt}{\lambda_n}$$

$$= \sum_{n=0}^{\infty} \frac{[\psi_n^*(1) - \alpha_o p(0)\mu(0)\psi_n^*(0)]\psi_n^*(\xi)}{\lambda_n} .$$

That is, according to (3.21) and (3.22), we have

$$\int_0^1 r(x)g(x,\xi)\mu(x)dx = g(1,\xi) - \alpha_o p(0)\mu(0)g(0,\xi) . \tag{4.12}$$

However, the application of the operator L to both sides of (4.12)

gives $\mu(\xi) = 0$ for $0 < \xi < 1$. This presents a contradiction because the right-hand side of (4.12) cannot be zero when α_o, as assumed, is an arbitrary positive quantity and $g(x,\xi) \not\equiv 0$. Perhaps it should be noted that (4.12) can be satisfied within the class of <u>generalized</u> functions if we take

$$\mu(x) = k_1\delta(x-1) + k_2\delta(x) \tag{4.13}$$

where k_1 and k_2 are constants, but this function cannot be admitted since it is not such that $\mu^2(x)$ is defined let alone integrable as the Riesz-Fischer theorem requires. We conclude that our supposition about $zG^*(1,1,z)$ is not tenable and therefore, for almost all z on C,

$$\lim_{\rho\to\infty} zG^*(1,1,z) = \infty \ . \tag{4.14}$$

We can now see from (4.7) that for almost all z on C

$$\lim_{\rho\to\infty} \mathcal{G}(t,\xi,z) = \lim_{\rho\to\infty} \left\{ G^*(t,\xi,z) - \frac{\alpha p(1) zG^*(t,1,z)G^*(\xi,1,z)}{1+\alpha p(1) zG^*(1.1.z)} \right\} \tag{4.15}$$

$$= \lim_{\rho\to\infty} G^*(t,\xi.z) - \lim_{\rho\to\infty} \frac{G^*(t,1,z)G^*(\xi,1,z)}{G^*(1.1,z)}$$

which is independent of α. Let us recall that we have already proved in Section 3 that if $\alpha = \alpha_1 \leq 0$ then

$$\lim_{\rho\to\infty} \frac{1}{2\pi i} \oint_C \frac{\mathcal{Y}(t,\xi,z)dz}{z} =^{o} 0$$

and

$$\lim_{\rho\to\infty} \frac{1}{2\pi i} \oint_C \frac{G^*(t,\xi,z)dz}{z} =^{o} 0 .$$

It follows from what has been demonstrated with respect to $zG^*(1,1,z)$ that

$$- \lim_{\rho\to\infty} \frac{1}{2\pi i} \oint_C \frac{\alpha p(1)G^*(t,1,z)G^*(\xi,1,z)dz}{1+\alpha p(1)zG^*(1,1,z)}$$

$$= - \lim_{\rho\to\infty} \frac{1}{2\pi i} \oint_C \frac{G^*(t,1,z)G^*(\xi,1,z)dz}{zG^*(1,1,z)} =^{o} 0 .$$

This implies that if $\alpha = -\alpha_1$ then

$$\lim_{\rho\to\infty} \frac{1}{2\pi i} \oint_C \frac{\mathcal{Y}(t,\xi,z)dz}{z} =^{o} 0$$

from which

$$\lim_{\rho\to\infty} \frac{1}{2\pi i} \oint_C \frac{\int_0^1 F(t)\mathcal{Y}(t,\xi,z)dtdz}{z} = 0 . \tag{4.16}$$

We have now shown that (4.16) is valid for all real values of α with $\alpha_o \geq 0$.

In preparation for remarks in the next paragraph it should be noted that by using the method given above we can show that when $\alpha = \alpha_1 \leq 0$ then for almost all z on C

$$\lim_{\rho \to \infty} z\mathcal{G}(0,0,z) = \infty$$

and

$$\lim_{\rho \to \infty} zG^*(0,0,z) = \infty \ .$$

Using these results with (4.14) the expression (4.7) shows that if $\alpha = \alpha_1 \leq 0$ then

$$-\lim_{\rho \to \infty} \frac{\alpha_1 p(1) z^2 [G^*(0,1,z)]^2}{1 + \alpha_1 p(1) z G^*(1,1,z)} = -\lim_{\rho \to \infty} \frac{z[G^*(0,1,z)]^2}{G^*(1,1,z)} = \infty \ .$$

From this it can be seen that when $\alpha = -\alpha_1$ we also have

$$\lim_{\rho \to \infty} z\mathcal{J}(0,0,z) = \infty \ .$$

With (4.16) established we can assign the role played by $G^*(x,\xi,z)$ to $\mathcal{J}(x,\xi,z)$ with $\alpha_o = 0$ and proceed to use the above method to show that, with no restriction on α_1 for $G(x,\xi,z)$, the restriction on α_o can also be removed. Thus we finally conclude that

$$\lim_{\rho \to \infty} \frac{1}{2\pi i} \oint_C \frac{\int_0^1 F(t)G(t,\xi,z)dtdz}{z} = 0 \tag{4.17}$$

holds for any real values of α_o and α_1. The general result (4.17) leads to the expansion

$$(4.18)\qquad f(\xi) = -\lim_{\rho\to\infty}\frac{1}{2\pi i}\oint_C Q[G(t,\xi,z),f(t)]dz$$

$$= -\sum_{n=0}^{\infty} Q\left[\frac{1}{2\pi i}\oint_{C_n} G(t,\xi,z)dz, f(t)\right]$$

provided $f(\xi)$ can be expressed in the form

$$f(\xi) = \int_0^1 F(x)g(x,\xi)dx$$

where $F(x)$ is such that

$$\lim_{\rho\to\infty}\int_0^1 F(x)\zeta_1(x,\xi)dx = 0 .$$

The function $\zeta_1(x,\xi)$ is

$$\zeta_1(x,\xi) = \frac{1}{2\pi i}\oint_C \frac{G(x,\xi,z)dz}{z}$$

which converges to a null function as $\rho \longrightarrow \infty$.

In order to illustrate our results let us return to the example in the introduction and use (4.18) to expand a twice differentiable function $f(x)$ in terms of the eigenfunctions defined by

$$(4.19)\qquad \psi_{xx} - \lambda\psi = 0 \qquad\qquad 0 < x < 1$$

$$(4.20)\qquad \psi_x(0) = 0$$

$$(4.21)\qquad \psi_x(1) = \alpha\lambda\psi(1) .$$

The eigenfunctions are

$$\psi_n(x) = a_n \cosh x\sqrt{\lambda_n} \ .$$

The eigenvalues $\{\lambda_n\}$ are the zeros of

$$\sqrt{\lambda} \sinh \sqrt{\lambda} = \alpha\lambda \cosh \sqrt{\lambda}$$

and these are real. The value $\lambda_o = 0$ is an eigenvalue and the corresponding eigenfunction is $\psi_o = a_o = \text{const} \neq 0$. If $\alpha \geq 1$ the eigenvalues, except λ_o, are all negative. If $\alpha < 1$ there is one positive eigenvalue. We will suppose that the eigenvalues are ordered with respect to absolute magnitude.

The Green's function associated with (4.19)-(4.21) is the function

$$G(x,\xi,\lambda) = \frac{\theta(x,\xi,\lambda)}{\omega(\lambda)}$$

which satisfies

(4.22) $$\theta_{xx} - \lambda\theta = \omega(\lambda)\delta(x-\xi)$$

(4.23) $$\theta_x(0,\xi,\lambda) = 0$$

(4.24) $$\theta_x(1,\xi,\lambda) = \alpha\lambda\theta(1,\xi,\lambda) \ .$$

The expansion formula for the present case is

$$(4.25)\qquad f(\xi) = -\lim_{\rho\to\infty}\frac{1}{2\pi i}\oint_C Q[G(t,\xi,z),f(t)]dz$$

$$= -\sum_{n=0}^{\infty}\frac{1}{2\pi i}\oint_{C_n} Q[G(t,\xi,z),f(t)]dz$$

where

$$Q[\Omega(t),\chi(t)] = \int_0^1 \Omega(t)\chi(t)dt - \alpha\Omega(1)\chi(1) .$$

From (2.27) we have

$$(4.26)\qquad \omega'(\lambda_n)\psi_n(\xi) = -Q[\psi_n,\psi_n]$$

$$= -\left[\int_0^1 \psi_n^2(x)dx - \alpha\psi_n^2(1)\right] .$$

We also have

$$(4.27)\qquad \lambda_n\omega'(\lambda_n)\psi_n(\xi) = -\left[\int_0^1 \psi_n(x)\psi_n''(x)dx - \alpha\lambda_n\psi_n^2(1)\right]$$

$$= \int_0^1 \psi_{nx}^2(x)dx .$$

The right-hand side of (4.27) is zero if and only if $\psi_n = \psi_o$. This means that each eigenvalue, with the possible exception of $\lambda_o = 0$, is a simple zero of $\omega(\lambda)$. The substitution of $n = 0$ in (4.26) gives

$$\omega'(0) = -(1-\alpha)\psi_o$$

which shows that $\lambda_o = 0$ is also a simple zero of $\omega(\lambda)$ provided $\alpha \neq 1$. Hence we see that (4.25) and (4.26) imply the following expansion for the case $\alpha \neq 1$:

$$(4.28) \qquad f(\xi) = \frac{\int_0^1 f(t)dt - \alpha f(1)}{1-\alpha} + \sum_{n=1}^{\infty} \frac{Q[f(t),\psi_n(t)]\psi_n(\xi)}{Q[\psi_n(t),\psi_n(t)]} \quad .$$

If $\alpha = 1$ then $\omega'(0) = 0$ which proves that λ_o is not a simple zero of $\omega(\lambda)$. Its multiplicity can be found by computing higher derivatives of $\omega(\lambda)$. These can be calculated from (4.22)-(4.24). It is not difficult to verify that

$$(4.29) \qquad \omega''(0) = \frac{2\psi_o}{3}$$

and

$$(4.30) \qquad \omega'''(0) = \frac{3}{2}\left[\frac{4\psi_o}{5} - \frac{2}{3}\psi_o\xi^2 + \frac{4b}{3}\right]$$

where b is a constant. In the process of finding these quantities we also find

$$\theta(x,\xi,0) = \psi_o$$

and

$$\theta_\lambda(x,\xi,0) = \psi_o \frac{x^2}{2} + b \quad .$$

Thus we see that $\lambda_o = 0$ is a double zero of $\omega(\lambda)$. Therefore the residue of $G(x,\xi,\lambda)$ at $\lambda = 0$ is

$$\left.\frac{d}{d\lambda}\frac{\lambda^2\theta}{\omega(\lambda)}\right|_{\lambda=0} = \frac{2\theta_\lambda(x,\xi,0)}{\omega''(0)} - \frac{2}{3}\frac{\omega'''(0)\theta(x,\xi,0)}{[\omega''(0)]^2}$$

$$= \frac{3}{2}(x^2+\xi^2) - \frac{9}{5}$$

when $\alpha = 1$. From the last result we find that for $\alpha = 1$:

$$f(\xi) = -Q[\{\frac{3}{2}(t^2+\xi^2) - \frac{9}{5}\}, f(t)] + \sum_{n=1}^{\infty} \frac{Q[f(t),\psi_n(t)]\psi_n(\xi)}{Q[\psi_n(t),\psi_n(t)]}$$

$$= \frac{3}{2}\left[f(1) - \int_0^1 t^2 f(t)dt\right] + \frac{3\xi^2}{2}\int_0^1 tf'(t)dt - \frac{9}{5}\int_0^1 tf'(t)dt$$

$$+ \sum_{n=1}^{\infty} \frac{Q[f(t),\psi_n(t)]\psi_n(\xi)}{Q[\psi_n(t),\psi_n(t)]} \quad .$$

It should be remarked that ξ^2 is not an eigenfunction of (4.19)-(4.21)!

More generally, if $\omega(z)$ has a zero of order k at $z = \lambda_n$ the corresponding term in the expansion (4.18) is

$$Q[\overline{\theta}_1(x,\xi,n), f(x)]$$

where $\overline{\theta}(x,\xi,n)/(z-\lambda_n)$ comes from the Laurent expansion of $G(x,\xi,z)$ for the neighborhood of $z = \lambda_n$. The function $\overline{\theta}_1$ can be obtained by substituting the expansion

$$G(x,\xi,z) = \sum_{\ell=-k}^{0} (z-\lambda_n)^\ell \overline{\theta}_{-\ell}(x,\xi,n) + \sum_{\ell=1}^{\infty} (z-\lambda_n)^\ell \theta_\ell(x,\xi,n)$$

in the equation

$$LG(x,\xi,z) - \lambda_n rG - (z-\lambda_n)rG = \delta(x-\xi) , \qquad 0 < x, \xi < 1$$

and the boundary conditions

$$G_x(0,\xi,z) + \beta_o G(0,\xi,z) = \alpha_o \lambda_n G(0,\xi,z) + \alpha_o (z-\lambda_n) G(0,\xi,z)$$

$$G_x(1,\xi,z) + \beta_1 G(1,\xi,z) = \alpha_1 \lambda_n G(1,\xi,z) + \alpha_1 (z-\lambda_n) G(1,\xi,z)$$

which define $G(x,\xi,z)$. We find from these equations, after equating coefficients of like powers of $(z-\lambda_n)$, that the barred quantities must satisfy

$$\overline{\theta}_k(x, \ ,n) = c\psi_n(x)$$

$$(L - \lambda_n r)\overline{\theta}_o = r\theta_1 + \delta(x-\xi)$$

$$(L - \lambda_n r)\overline{\theta}_1 = r\overline{\theta}_{j+1} , \qquad j = 1,2,\ldots,(k-1)$$

with the boundary conditions

$$\overline{\theta}_{jk}(0,\xi,n) + \beta_o \overline{\theta}_j(0,\xi,n) = \alpha_o \lambda_n \overline{\theta}_j(0,\xi,n) + \alpha_o \overline{\theta}_{j+1}(0,\xi,n)$$

$$j = 0,1,\ldots,(k-1) .$$

$$\overline{\theta}_{jx}(1,\xi,n) + \beta_1 \overline{\theta}_j(1,\xi,n) = \alpha_1 \lambda_n \overline{\theta}_j(1,\xi,n) + \alpha_1 \overline{\theta}_{j+1}(1,\xi,n)$$

The above equations hold only if the functions $\overline{\theta}_j$ satisfy the compatibility conditions

$$Q[\psi_n, \overline{\theta}_{j+1}] = 0 , \qquad j = 1,2,\ldots,(k-1)$$

$$-Q[\psi_n, \overline{\theta}_1] = \psi_n(\xi) .$$

We can see from the foregoing that $\overline{\theta}_1(x,\xi)$ is not an eigenfunction (if $k > 1$) but must satisfy

$$\{\frac{1}{r}(L - \lambda_n r)\}^{k-1} \overline{\theta}_1(x,\xi,n) = c\psi_n(x) .$$

For this case, $\overline{\theta}_1$ is called a <u>generalized eigenfunction</u>.

5. Generalization of a Fredholm Equation

We saw in Section 2 that $G(x,\xi,\lambda)$ satisfies the integral equation

$$(5.1)\quad G(t,\xi,z) - G(t,\xi,\sigma) = (z-\sigma)\left\{\begin{array}{l} \displaystyle\int_0^1 r(x)G(x,t,z)G(x,\xi,\sigma)dx \\ \\ \quad - p(1)\alpha_1 G(1,t,z)G(1,\xi,\sigma) \\ \\ \quad + p(0)\alpha_0 G(0,t,z)G(0,\xi,\sigma) \end{array}\right\} .$$

If we use the substitutions

$$G(t,\xi,\lambda) = \frac{H(t,\xi,\lambda)}{\sqrt{r(t)r(\xi)}} \quad ,$$

$$\alpha_1 = \frac{r(1)a_1}{p(1)} \quad , \qquad \alpha_o = \frac{r(0)a_o}{p(0)} \quad ;$$

equation (5.1) becomes

$$(5.2)\quad H(t,\xi,z) - H(t,\xi,\sigma) = (z-\sigma)\left\{\begin{array}{l} \displaystyle\int_0^1 H(x,t,z)H(x,\xi,\sigma)dx \\ \\ \quad - a_1 H(1,t,z)H(1,\xi,\sigma) \\ \\ \quad + a_o H(0,t,z)H(0,\xi,\sigma) \end{array}\right\} \quad .$$

If we set $z = \lambda$; $\sigma = 0$ and then $z = 0$; $\sigma = \lambda$ in (5.2) we have

$$H(t,\xi,\lambda) = H(t,\xi,0) + \lambda\left\{\begin{array}{l} \displaystyle\int_0^1 H(x,\xi,0)H(x,t,\lambda)dx \\ \\ \quad - a_1 H(1,\xi,0)H(1,t,\lambda) \\ \\ \quad + a_o H(0,\xi,0)H(0,t,\lambda) \end{array}\right\}$$

(5.3)

$$= H(t,\xi,0) + \lambda\left\{\begin{array}{l} \displaystyle\int_0^1 H(x,t,0)H(x,\xi,\lambda)dx \\ \\ \quad - a_1 H(1,t,0)H(1,\xi,\lambda) \\ \\ \quad + a_o H(0,t,0)H(0,\xi,\lambda) \end{array}\right\} \quad ,$$

The last equations form a particular case of the more general equations

$$R(t,\xi,\lambda) = K(t,\xi) + \lambda \left\{ \begin{array}{l} \displaystyle\int_0^1 K(x,\xi)R(x,t,\lambda)dx \\ \\ \quad - a_1 K(1,\xi)R(1,t,\lambda) \\ \\ \quad + a_0 K(0,\xi)R(0,t,\lambda) \end{array} \right\}$$

(5.4)

$$= K(t,\xi) + \lambda \left\{ \begin{array}{l} \displaystyle\int_0^1 K(x,t)R(x,\xi,\lambda)dx \\ \\ \quad - a_1 K(1,t)R(1,\xi,\lambda) \\ \\ \quad + a_0 K(0,t)R(0,\xi,\lambda) \end{array} \right\} ,$$

where $K(x,\xi) = K(\xi,x)$ is a symmetric real kernel. The function $R(t,\xi,\lambda)$ which satisfies (5.4) is the resolvent kernel for the equation

$$\phi(t) = w(t) + \lambda \left\{ \begin{array}{l} \displaystyle\int_0^1 K(x,t)\phi(x)dx \\ \\ \quad - a_1 K(1,t)\phi(1) \\ \\ \quad + a_0 K(0,t)\phi(0) \end{array} \right\} . \tag{5.5}$$

That is, if λ is not an eigenvalue, the solution of (5.5) can be expressed as

$$\phi(\xi) = w(\xi) + \lambda \left\{ \begin{array}{l} \displaystyle\int_0^1 w(t)R(t,\xi,\lambda)dt \\ \\ - a_1 w(1)R(1,\xi,\lambda) \\ \\ + a_o w(0)R(0,\xi,\lambda) \end{array} \right\} . \tag{5.6}$$

When $a_1 = a_o = 0$, equation (5.5) reduces to the ordinary Fredholm integral equation of the second kind, namely

$$\phi(t) = w(t) + \lambda \int_0^1 K(x,t)\phi(x)dx , \tag{5.7}$$

with the resolvent kernel $R_o(t,\xi,\lambda)$ which must satisfy

$$\begin{aligned} R_o(t,\xi,\lambda) &= K(t,\xi) + \lambda \int_0^1 K(x,\xi)R_o(x,t,\lambda)dx \\ &= K(t,\xi) + \lambda \int_0^1 K(x,t)R_o(x,\xi,\lambda)dx . \end{aligned} \tag{5.8}$$

Let us suppose that the continuous symmetric kernel $K(x,t)$ is such that there exists a complete set of eigenfunctions, $\{\psi_n(x)\}$; and eigenvalues $\{\lambda_n\}$ which satisfy

$$\psi_n(t) = \lambda_n \int_0^1 K(x,t)\psi_n(x)dx . \tag{5.9}$$

A necessary and sufficient condition for completeness is that

$$\int_0^1 K(x,t)\chi(x)dx = 0 \tag{5.10}$$

implies $\chi(x) =^{0} 0$. The eigenvalues are real; and the eigenfunctions constitute an orthogonal set which we assume is normalized. Then, according to the Hilbert-Schmidt theory of (5.7), if f(x) can be represented in the form

$$f(x) = \int_0^1 K(t,x)F(t)dt \tag{5.11}$$

where F(t) is continuous, we have the expansion

$$f(x) = \sum_{n=1}^{\infty} \psi_n(x) \int_0^1 f(t)\psi_n(t)dt \tag{5.12}$$

and the series is uniformly and absolutely convergent on the closed interval [0,1]. Consequently, the resolvent kernel possesses the expansion

$$R_o(t,\xi,\lambda) = K(t,\xi) - \lambda \sum_{n=1}^{\infty} \frac{\psi_n(\xi)\psi_n(t)}{\lambda_n(\lambda - \lambda_n)} . \tag{5.13}$$

With this, simple calculations show that (5.12) can be presented in the form

$$\lim_{\rho \to \infty} \frac{1}{2\pi i} \oint_C \frac{\int_0^1 F(t)R_o(t,x,z)dtdz}{z} = 0 . \tag{5.14}$$

It can be shown that the resolvent kernel $R(t,\xi,\lambda)$ is a ratio

$$R(x,\xi,\lambda) = \frac{\theta(x,\xi,\lambda)}{\omega(\lambda)} \tag{5.15}$$

of two entire functions of λ. The zeros of $((\lambda)$ are just the eigenvalues of

$$\psi(t) = \lambda \left\{ \begin{array}{l} \int_0^1 K(x,t)\psi(x)dx \\ \\ - a_1 K(1,t)\psi(1) \\ \\ + a_0 K(0,t)\psi(0) \end{array} \right\} . \tag{5.16}$$

Starting with (5.15) and using the method of Section 3 we can prove that

$$\lim_{\rho \to \infty} \frac{1}{2\pi i} \oint_C \frac{\int_0^1 F(t)R(t,x,z)dtdz}{z} = 0 \tag{5.17}$$

provided $a_o \geq 0$ and $a_1 \leq 0$. Furthermore, these restrictions on a_o and a_1 with respect to sign can be removed by proceeding in essentially the same manner as we did in Section 4.

In order to indicate briefly how the above mentioned restrictions on a_o and a_1 can be removed consider the resolvent kernel $\mathcal{R}(t,\xi,\lambda) = \mathcal{R}(\xi,t,\lambda)$ which satisfies

$$\mathcal{R}(t,\xi,\lambda) = K(t,\xi) + \lambda \left\{ \begin{array}{l} \int_0^1 K(x,t)\,\mathcal{R}(x,\xi,\lambda)dx \\ \\ - a\,K(1,t)\,\mathcal{R}(1,\xi,\lambda) \\ \\ + a_o K(0,t)\,\mathcal{R}(0,\xi,\lambda) \end{array} \right\} \tag{5.18}$$

$$= K(t,\xi) + \lambda Q_1[K(x,t), \mathcal{R}(x,\xi,\lambda)]$$

where $a_o \geq 0$; and a is to be assigned the values a_1 and $-a_1$ where $a_1 \leq 0$. Introduce $R^*(t,\xi,\lambda) = R^*(\xi,t,\lambda)$ defined by

$$(5.19) \qquad R^*(t,\xi,\lambda) = K(t,\xi) + \lambda \left\{ \begin{array}{l} \int_0^1 K(x,t)R^*(x,\xi,\lambda)dx \\ \\ + a_o K(0,t)R^*(0,\xi,\lambda) \end{array} \right\}$$

$$= K(t,\xi) + \lambda Q_o[K(x,t),R^*(x,\xi,\lambda)] \ .$$

The resolvent $\mathcal{R}(t,\xi,\lambda)$ can be presented in terms of $R^*(t,\xi,\lambda)$ as follows:

$$\lambda Q_o[R^*(t,s,\lambda), \mathcal{R}(t,\xi,\lambda)]$$

$$= \lambda Q_o[K(t,\xi),R^*(t,s,\lambda)] + \lambda^2 Q_1\Big[Q_o[K(t,x),R^*(t,s,\lambda)], \mathcal{R}(x,\xi,\lambda)\Big]$$

$$= \lambda Q_o[K(t,\xi),R^*(t,s,\lambda)] + \lambda Q_1[\{R^*(x,s,\lambda) - K(x,s)\}, \mathcal{R}(x,\xi,\lambda)]$$

$$= \lambda Q_o[K(t,\xi),R^*(t,s,\lambda)] + \lambda Q_o[R^*(x,s,\lambda), \mathcal{R}(x,\xi,\lambda)]$$

$$- aR^*(1,s,\lambda)\mathcal{R}(1,\xi,\lambda) - \lambda Q_1[K(x,s), \mathcal{R}(x,\xi,\lambda)]$$

$$\lambda Q_1[K(x,s), \mathcal{R}(x,\xi,\lambda)] = R^*(\xi,s,\lambda) - K(\xi,s) - aR^*(1,s,\lambda)\mathcal{R}(1,\xi,\lambda) \ .$$

The substitution of the last result in (5.18) yields

$$\mathcal{R}(t,\xi,\lambda) = R^*(t,\xi,\lambda) - aR^*(1,t,\lambda)\mathcal{R}(1,\xi,\lambda)$$

and from this we find

$$\mathcal{R}(t,\xi,\lambda) = R^*(t,\xi,\lambda) - \frac{aR^*(t,1,\lambda)R^*(\xi,1,\lambda)}{1+aR^*(1,1,\lambda)}$$

the analogue of (4.7). After this, the procedure for the removal of the restrictions on a_o and a_1 can be developed in accordance with the ideas used in Section 4 subsequent to equation (4.7). In the course of proving

$$\lim_{\rho\to\infty} zR^*(1,1,z) = \infty$$

we would use the assumption (5.10) at a point corresponding approximately to equation (4.12).

The end result is the formula

$$\lim_{\rho\to\infty} \frac{1}{2\pi i} \oint_C \frac{\int_0^1 F(t)R(t,x,z)dtdz}{z} = 0 \tag{5.20}$$

valid for all real a_o and a_1 when $\int_0^1 F^2(t)dt$ exists. If we set

$$f(x) = \int_0^1 K(t,x)F(t)dt$$

then (5.20) generates the expansion

$$f(x) = - \lim_{\rho \to \infty} \frac{1}{2\pi i} \oint_{C} Q_1[f(s),R(s,x,z)]dz$$

$$= - \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{C_n} Q_1[f(s),R(s,x,z)]dz \quad .$$

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1. ORIGINATING ACTIVITY (Corporate author)
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2a. REPORT SECURITY CLASSIFICATION
not classified

2b GROUP
none

3. REPORT TITLE
Residue Expansions for Certain Green's Functions and Resolvent Kernels

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Technical Report April 1968

5. AUTHOR(S) (Last name, first name, initial)
Peters, Arthur S.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
April 1968	48	7

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
Nonr-285(55)	IMM 366
b. PROJECT NO. NR 062-160	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	none

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11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
none	U.S. Navy, Office of Naval Research 207 West 24th St., New York, N.Y.

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$$\frac{d}{dx}\, p(x)\, \frac{d}{dx}\, \psi(x,\lambda) + q(x)\psi - \lambda r(x)\psi = 0 , \qquad 0 < x < 1 ;$$

subject to the boundary conditions

$$\psi_x(0,\lambda) + \beta_o \psi(0,\lambda) = \alpha_o \lambda \psi(0,\lambda) ,$$

$$\psi_x(1,\lambda) + \beta_1 \psi(1,\lambda) = \alpha_1 \lambda \psi(1,\lambda) .$$

The expansions stem from the Poincaré-Birkhoff formula. It is shown that this formula can be extended so as to yield expansions in terms of the eigenfunctions of

$$\psi(t) = \lambda[a_o K(0,t)\psi(0) - a_1 K(1,t)\psi(1) + \int_0^1 K(t,x)\psi(x)dx]$$

where $K(t,x) = K(x,t)$. The quantities α_o, α_1, a_o and a_1 are supposed to be real but they are otherwise not restricted.

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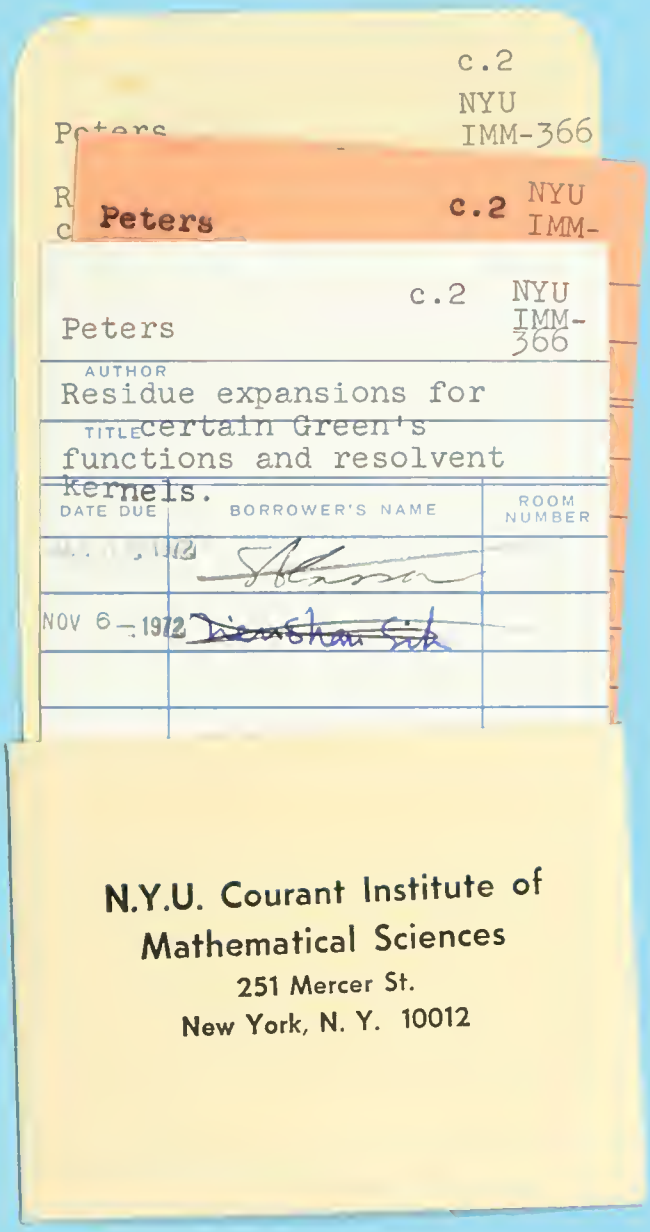
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